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Exact analytical solutions for bending of rectangular plates with a partial internal line support

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Abstract This paper studies the behavior of uniformly loaded rectangular thin plates with a partial internal line support. The highlight of the problem is that the analytical formulation explicitly considers the moment singularities that occur at the tips of partial internal line supports. The proper finite Hankel transform is used to transform a pair of dual-series equations obtained from the mixed conditions along the partial internal line support to a single Fredholm integral equation. Numerical results concerning deflection, bending moment, resultant forces, and bending-stress intensity factors are given for a square plate. Some results are also compared with the case of a square plate without partial internal line support.

Keywords Dual-series equations · Fredholm integral equation · Hankel transform · Moment singularities · Plate-bending

1 Introduction

In problems concerning a finite plate with mixed boundary conditions, Keer and Sve [1] used a solution technique similar to that of Westmann and Yang [2] to analyze the bending of cracked rectangular plates subjected to a uniformly distributed load. The finite Hankel integral-transform method was used, and the singularity in the bending moments [3,4] that is proportional to the inverse square root of the distance from the root of the crack was also explicitly introduced in the analysis. Since the moment singularities are of an inverse-square-root type, it turns out that the supplemented or Kirchhoff shearing force becomes nonintegrable. The extension of this method has allowed to solve vibration and buckling problems for rectangular plates with cracks and rectangular plates with mixed edge conditions as done by Stahl and Keer [5] and Keer and Stahl [6], respectively. In each problem the singular part of the solution was isolated and treated analytically, and the problem was reduced to determining the eigenvalues of a homogeneous Fredholm integral equation of the second kind.

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In addition, Kiattikomol et al. [7] extended the method to bending of rectangular plates that are simply supported on two opposite edges, but may only be partially constrained along the other two edges. In the same manner of [7], rectangular plates that involve considerations of advancing contact were investigated by Dundurs et al. [8] where no singularity in the moments is allowed. However, the shear singularity that existed at the ends of a contact interval is still found to be of inverse-square-root type. Therefore, the shear distribution along the supports is integrable and integration of all the support reactions showed that the total load is balanced.

As mentioned in the previous works [1,5–8], the method of solution can be summarized as follows: all problems studied are first formulated through the dual-series equations, while taking advantage of the proper finite Hankel-transform techniques and satisfying the nature of the singularities; then, the dual-series equations can further be reduced to a Fredholm integral equation governing the problem solution and solved numerically for an unknown auxiliary function by using standard numerical methods.

Consider the bending of a simply supported rectangular plate with a partial internal line support and laterally loaded by a general loading, which was investigated by Yang [9]. In his work, the problem was reduced to the solution of a singular integral equation governing the pressure distribution along the partial internal line support by means of a finite Hilbert transform. It is worth noticing that the pressure distribution along the partial internal line support is square-root singular. This result does not agree with the work of Williams [4], who pointed out that the moment singularity should exist outside of the partial internal-line-support region and the shear distribution along the support is nonintegrable. Subsequently, Stahl and Keer [10] studied the free vibration and buckling of a rectangular plate with a centrally located internal line support by using the method of finite Hankel integral transforms. In their work, the correct singularity in the moment was assumed to be located at the tips of the internal support. Recently, Sompornjaroensuk and Kiattikomol [11] examined the free contact [12] between rectangular plates and a sagged partial internal line support. The problems were formulated through dual-series equations similar to those used by Dundurs et al. [8], and the correct local inverse-square-root singularity in the shear that corresponded with the nature of the contact behavior was considered in the formulation.

The aims of the present investigation are (1) to reconsider the bending problem of a simply supported rectangular plate with a partial internal line support, which has been treated by Yang [9], and (2) to consider the problems of a rectangular plate simply supported on two opposite edges and the remaining edges having the same type of support, either clamped or free, and having a partial internal line support along the plate center line. All plate problems involve a uniformly distributed load. The mixed boundary conditions are treated by introducing the finite Hankel integral transform [11], providing the correct square-root moment singularity [4] at the tips of the partial internal line support. The deflections, bending moments, corner forces, and the proportion of the total applied force carried by the supports of the plate are given. The bending-stress intensity factors near the tip of a partial internal line support of the plate are also determined.

2 Theoretical formulation

For a uniformly loaded rectangular plate, the length of a partial internal line support located at the center of the plate can be varied symmetrically as indicated in Fig. 1. Let a, $2\bar{b}$ be the actual length and actual width of the plate, respectively, which are scaled by the factor π/a. Thus, the governing equation of the plate in scaled coordinates (x, y) can be expressed as [13, Art. 21]

$$\Delta\Delta w = \frac{qa^4}{\pi^4 D}, \tag{1}$$

where

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \tag{2}$$

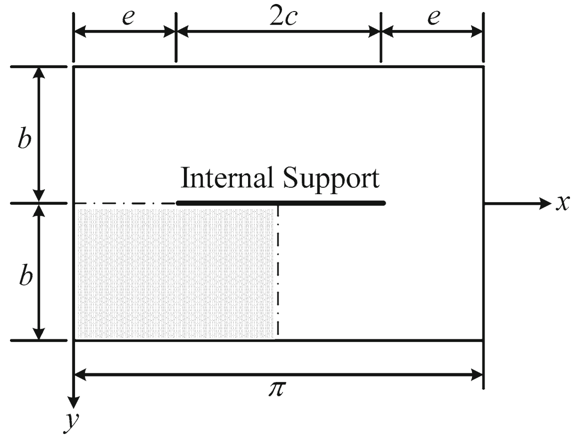

Fig. 1 Plate with a partial internal line support

in which Δ is the two-dimensional Laplacian operator, w is the deflection of the plate, q is the uniformly distributed load, and D is the plate rigidity defined as

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{3}$$

where E is Young's modulus, ν is Poisson's ratio, and h is the plate thickness.

Since the edges of the plate at $x = 0, \pi$ are simple supports, the deflection function that satisfies the governing equation in (1) can be assumed to be of the form

$$w(x, y) = \frac{qa^4}{D} \sum_{m=1,3,5,\ldots}^{\infty} \left[\frac{4}{\pi^5 m^5} + Y_m(y) \right] \sin(mx), \tag{4}$$

with

$$Y_m(y) = A_m^{(i)} \cosh(my) + B_m^{(i)} my \sinh(my) + C_m^{(i)} \sinh(my) + D_m^{(i)} my \cosh(my), \tag{5}$$

where $A_m^{(i)}$, $B_m^{(i)}$, $C_m^{(i)}$, and $D_m^{(i)}$ are unknown constants to be determined from the boundary and internal constraint conditions. The superscript (i) refers to the three cases of the plate having different support conditions at $|y| = b$ namely, simple ($i = 1$), clamped ($i = 2$), and free ($i = 3$) supports.

Due to the two-fold symmetry of the deflection function, only the region of the plate bounded by $0 \le x \le \pi/2$ and $0 \le y \le b$ in Fig. 1 is considered. The mixed boundary conditions at $0 \le x \le \pi/2$ and $y = 0$ along the partial internal line support are specified as follows:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{for } e < x \le \pi/2, \tag{6,7,8}$$

$$\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0 \quad \text{for } 0 \le x < e. \tag{9}$$

Equation (9) is the condition of Kirchhoff shear.

It is noted that Eqs. (8) and (9) are used in formulating the dual series. Substitution of (4) in (8)–(9) and applying the internal constraint condition on the slope $\partial w/\partial y = 0$, $y = 0$ due to the symmetry of the deflection function together with the boundary conditions at $y = b$, leads to the dual-series equations

$$\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(i)} \sin(mx) = 0; \quad e < x \le \pi/2, \tag{10}$$

$$\sum_{m=1,3,5,\ldots}^{\infty} m^3 P_m^{(i)}(1+F_m^{(i)})\sin(mx) = \sum_{m=1,3,5,\ldots}^{\infty} G_m^{(i)}\sin(mx); \quad 0 \le x < e, \tag{11}$$

where the definitions of the unknown function $P_m^{(i)}$, the weight function $F_m^{(i)}$, the known function $G_m^{(i)}$, and the unknown constants in (5) can be defined separately according to each plate problem as follows:

(I) Simply supported edges at $|y| = b$ and $i = 1$;

$$P_m^{(1)} = \frac{2}{\pi^5 m^5}\left[2 - \frac{(2+\beta\tanh\beta)}{\cosh\beta}\right] + D_m^{(1)}\frac{(\sinh\beta\cosh\beta - \beta)}{\cosh^2\beta}, \tag{12}$$

$$1 + F_m^{(1)} = \frac{\cosh^2\beta}{\sinh\beta\cosh\beta - \beta}, \tag{13}$$

$$G_m^{(1)} = \frac{2}{\pi^5 m^2}\left[2 - \frac{(2+\beta\tanh\beta)}{\cosh\beta}\right]\left[\frac{\cosh^2\beta}{\sinh\beta\cosh\beta - \beta}\right], \tag{14}$$

$$A_m^{(1)} = -\frac{2(2+\beta\tanh\beta)}{\pi^5 m^5\cosh\beta} + D_m^{(1)}\frac{(\sinh\beta\cosh\beta - \beta)}{\cosh^2\beta}, \tag{15}$$

$$B_m^{(1)} = \frac{2}{\pi^5 m^5\cosh\beta} - D_m^{(1)}\tanh\beta, \quad \beta = mb, \tag{16,17}$$

(II) Clamped edges at $|y| = b$ and $i = 2$;

$$P_m^{(2)} = \frac{4\Delta_3}{\pi^5 m^5} + D_m^{(2)}\frac{\Delta_2}{\Delta_1}, \quad 1 + F_m^{(2)} = \frac{\Delta_1}{\Delta_2}, \quad G_m^{(2)} = \frac{4\Delta_1\Delta_3}{\pi^5 m^2\Delta_2}, \tag{18,19,20}$$

$$A_m^{(2)} = -\frac{4\sinh\beta(1+\beta\coth\beta)}{\pi^5 m^5(\beta + \sinh\beta\cosh\beta)} + D_m^{(2)}\frac{\Delta_2}{\Delta_1}, \tag{21}$$

$$\Delta_1 B_m^{(2)} = \frac{4}{\pi^5 m^5\cosh\beta} - D_m^{(2)}\tanh\beta, \tag{22}$$

$$\Delta_1 = 1 - \beta(\tanh\beta - \coth\beta), \quad \Delta_2 = \tanh\beta + \beta^2(\tanh\beta - \coth\beta), \tag{23,24}$$

$$\Delta_3 = \frac{(1-\cosh\beta)(\beta - \sinh\beta)}{\beta + \sinh\beta\cosh\beta}, \tag{25}$$

(III) Free edges at $|y| = b$ and $i = 3$;

$$P_m^{(3)} = \frac{4\bar{\Delta}_3}{\pi^5 m^5\bar{\Delta}_1} + D_m^{(3)}\frac{\bar{\Delta}_2}{\bar{\Delta}_1}, \quad 1 + F_m^{(3)} = \frac{\bar{\Delta}_1}{\bar{\Delta}_2}, \quad G_m^{(3)} = \frac{4\bar{\Delta}_3}{\pi^5 m^2\bar{\Delta}_2}, \tag{26,27,28}$$

$$\bar{\Delta}_1 A_m^{(3)} = -\frac{4\nu(\beta\cosh\beta - \eta''\sinh\beta)}{\pi^5 m^5} + D_m^{(3)}\bar{\Delta}_2, \tag{29}$$

$$\bar{\Delta}_1 B_m^{(3)} = \frac{4\nu \sinh\beta}{\pi^5 m^5} - D_m^{(3)}[2 + (3+\nu)\sinh^2\beta], \tag{30}$$

$$\bar{\Delta}_1 = (3+\nu)\sinh\beta\cosh\beta - (1-\nu)\beta, \tag{31}$$

$$\bar{\Delta}_2 = (1-\nu)\beta^2 + (1+\nu)\eta'' + (3+\nu)\cosh^2\beta, \quad \eta'' = (1+\nu)/(1-\nu), \tag{32,33}$$

$$\bar{\Delta}_3 = (3+\nu)\sinh\beta\cosh\beta - (1-\nu)\beta + \nu(\eta''\sinh\beta - \beta\cosh\beta). \tag{34}$$

For all cases, the constant $C_m^{(i)}$ can be taken as

$$C_m^{(i)} = -D_m^{(i)}; \; i = 1, 2, 3. \tag{35}$$

Equation (10) and also Eqs. (6) and (7) can be automatically satisfied by choosing the unknown function $P_m^{(i)}$ in the form of a finite Hankel integral transform (see [7])

$$m^2 P_m^{(i)} = \int_0^e t\varphi^{(i)}(t)\left[J_1(mt) - \frac{t}{e}J_1(me)\right]\mathrm{d}t; \quad m = 1, 3, 5, \ldots, \tag{36}$$

where $\varphi^{(i)}(\cdot)$ is an unknown auxiliary function and $J_n(\cdot)$ is the Bessel function of the first kind and order n. The choice of $P_m^{(i)}$ in (36) has to exhibit a square-root moment singularity [4] at the tips of the partial internal line support. The verifications can easily be made by using the identities given in [6,7,10].

Substituting the unknown function $P_m^{(i)}$ in (36) in (11) and using the integral representation of series involving Bessel function [10]

$$\sum_{m=1,3,5,\ldots}^{\infty} J_1(mt)\cos(mx) = \frac{1}{2}t^{-1} - \frac{1}{2}xt^{-1}(x^2-t^2)^{-1/2}H(x-t) + \int_0^\infty [\exp(\pi s)+1]^{-1} I_1(ts)\cosh(xs)\mathrm{d}s; \quad x + t < \pi, \tag{37}$$

where $H(\cdot)$ is the Heaviside unit step function and $I_n(\cdot)$ is the modified Bessel function of the first kind and order n, one may write Eq. (11) in the form of Abel's integral equation as

$$\int_0^x \frac{x\varphi^{(i)}(t)}{\sqrt{x^2-t^2}}\mathrm{d}t = h(x); \quad 0 \le x < e, \tag{38}$$

and the solution of $\varphi^{(i)}(t)$ is

$$\varphi^{(i)}(t) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{h(x)}{\sqrt{t^2-x^2}}\mathrm{d}x; \quad 0 \le t < e, \tag{39}$$

where

$$\begin{aligned} h(x) = e\int_0^1 \varphi^{(i)}(er)\left\{1 - r^2 + 2er\int_0^\infty [\exp(\pi s)+1]^{-1}[I_1(ser) - rI_1(se)]\cosh(xs)\mathrm{d}s\right\}\mathrm{d}r \\ +2e^2\int_0^1 r\varphi^{(i)}(er)\sum_{m=1,3.5,\ldots}^{\infty} F_m^{(i)}[J_1(mer) - rJ_1(me)]\cos(mx)\mathrm{d}r \\ -2\sum_{m=1,3,5,\ldots}^{\infty} m^{-1}G_m^{(i)}\cos(mx); \quad 0 \le x < e. \end{aligned} \tag{40}$$

Utilizing the identities found in Gradshteyn and Ryzhic [14, pp. 426–427], one may evaluate the right-hand side of (39) in closed-form. The final result is an inhomogeneous Fredholm integral equation of the second kind,

$$\Psi^{(i)}(\rho) + \int_0^1 K(\rho, r)\Psi^{(i)}(r)\mathrm{d}r = f(\rho); \quad 0 \le \rho \le 1, \tag{41}$$

where $t = er, e\rho$.

$$\Psi^{(i)}(\rho) = \varphi^{(i)}(e\rho), \quad \Psi^{(i)}(r) = \varphi^{(i)}(er), \tag{42,43}$$

$$K(\rho, r) = 2e^2 r \sum_{m=1,3,5,\ldots}^{\infty} m F_m^{(i)} [J_1(mer) - r J_1(me)] J_1(me\rho) \\ -2e^2 r \int_0^{\infty} s[\exp(\pi s) + 1]^{-1} [I_1(ser) - r I_1(se)] I_1(se\rho) \mathrm{d}s, \tag{44}$$

$$f(\rho) = 2 \sum_{m=1,3,5,\ldots}^{\infty} G_m^{(i)} J_1(me\rho). \tag{45}$$

The functions $F_m^{(i)}$ and $G_m^{(i)}$ have been defined previously for $i = 1, 2, 3$ in Eqs. (13), (19), (27) and Eqs. (14), (20), (28), respectively. The Fredholm integral equation in (41) can be solved numerically to determine the auxiliary function $\Psi^{(i)}(\rho)$. Once the $\Psi^{(i)}(\rho)$ is known, $P_m^{(i)}$ in (36) can be determined and then $D_m^{(i)}$ in (12), (18), (26), $A_m^{(i)}$ in (15), (21), (29), $B_m^{(i)}$ in (16), (22), (30) and $C_m^{(i)}$ in (35) can also be obtained. Therefore, the deflection function in (4) and the other quantities of plate can be determined. It is noted that Eq. (41) cannot be solved for $e/\pi \to 0.5$ because the improper infinite integral in the kernel $K(\rho, r)$ fails to converge.

For plates without a partial internal line support, $C_m^{(i)}$ and $D_m^{(i)}$ in $Y_m(y)$ of (5) have to be zero due to the symmetry of the deflection function. Thus, the remaining unknown constants $A_m^{(i)}$ and $B_m^{(i)}$ can be redefined as

(I) Simply supported edges at $|y| = b$ and $i = 1$, no internal support;

$$A_m^{(1)} = -\frac{2(2 + \beta \tanh \beta)}{\pi^5 m^5 \cosh \beta}, \quad B_m^{(1)} = \frac{2}{\pi^5 m^5 \cosh \beta}, \tag{46,47}$$

(II) Clamped edges at $|y| = b$ and $i = 2$, no internal support;

$$A_m^{(2)} = -\frac{4 \sinh \beta (1 + \beta \coth \beta)}{\pi^5 m^5 (\beta + \sinh \beta \cosh \beta)}, \quad \Delta_1 B_m^{(2)} = \frac{4}{\pi^5 m^5 \cosh \beta}, \tag{48,49}$$

(III) Free edges at $|y| = b$ and $i = 3$, no internal support;

$$\bar{\Delta}_1 A_m^{(3)} = -\frac{4\nu(\beta \cosh \beta - \eta'' \sinh \beta)}{\pi^5 m^5}, \quad \bar{\Delta}_1 B_m^{(3)} = \frac{4\nu \sinh \beta}{\pi^5 m^5}, \tag{50,51}$$

where Δ_1 and $\bar{\Delta}_1$ are unchanged as defined by (23) and (31), respectively.

3 Physical quantities

The general deflections in (4), the deflections and bending moments along the line outside the partial internal line support ($0 \leq x \leq e$, $y = 0$) and along the normal to the partial internal line support ($x = \pi/2$, $0 \leq y \leq b$), and the corner forces are determined. For plates without a partial internal line support, all physical quantities can easily be determined due to the absence of a singularity in the solutions.

To determine the deflection $w(x, 0)$, substitute $y = 0$ in (4), use the unknown constants $A_m^{(i)}$, $B_m^{(i)}$, $C_m^{(i)}$, $D_m^{(i)}$, the unknown function $P_m^{(i)}$ in (36) for each of the cases $i = 1, 2, 3$ and the identity [10]

$$\sum_{m=1,3,5,\ldots}^{\infty} m^{-2} J_1(mt) \sin(mx) = \left\{ \begin{array}{ll} \frac{1}{4}\left[\frac{x}{t}(t^2 - x^2)^{1/2} + t \sin^{-1}\left(\frac{x}{t}\right)\right] & : x < t \\ \frac{\pi}{8} t & : x \geq t \end{array} \right\}; \quad x + t < \pi; \tag{52}$$

then, the deflection can be reduced to the form

$$\frac{w(x, 0)}{(qa^4/D)} = \frac{\pi e^3}{8} \int_0^{\xi} \rho^2 \Psi^{(i)}(\rho) \mathrm{d}\rho + \frac{e^3}{4} \int_{\xi}^1 \left[\xi \sqrt{\rho^2 - \xi^2} + \rho^2 \sin^{-1}(\xi/\rho)\right] \Psi^{(i)}(\rho) \mathrm{d}\rho \\ -\frac{e^3}{4} \int_0^1 \rho^2 \left[\xi \sqrt{1 - \xi^2} + \sin^{-1} \xi\right] \Psi^{(i)}(\rho) \mathrm{d}\rho, \tag{53}$$

where $\xi = x/e$ and $0 \le \xi, \rho \le 1$.

For the deflection $w(\pi/2, y)$ along the center line normal to the partial internal line support, setting $x = \pi/2$ and substituting in (4), yields

$$\frac{w(\pi/2, y)}{(qa^4/D)} = \sum_{m=1,3,5,\ldots}^{\infty} \{4/(\pi m)^5 + A_m^{(i)} \cosh(my) + B_m^{(i)} my \sinh(my) + C_m^{(i)}[\sinh(my) - my\cosh(my)]\}(-1)^{(m-1)/2}; \quad 0 \le y \le b, \tag{54}$$

in which $A_m^{(i)}$, $B_m^{(i)}$, $C_m^{(i)}$, and $D_m^{(i)}$ are expressed in an integral-representation form and in terms of $F_m^{(i)}$ and $G_m^{(i)}$ as the following relations:

(I) Simply supported edges at $|y| = b$ and $i = 1$;

$$B_m^{(1)} = \frac{2\mathrm{sech}\beta}{\pi^5 m^5} + \tanh\beta \left\{ \frac{G_m^{(1)}}{m^3} - (1 + F_m^{(1)}) \left(\frac{e}{m}\right)^2 \int_0^1 \rho \Psi^{(1)}(\rho)[J_1(me\rho) - \rho J_1(me)]\mathrm{d}\rho \right\}, \tag{55}$$

(II) Clamped edges at $|y| = b$ and $i = 2$;

$$B_m^{(2)} = \frac{4\mathrm{sech}\beta}{\pi^5 m^5 \Delta_1} + \frac{\tanh\beta}{\Delta_1} \left\{ \frac{G_m^{(2)}}{m^3} - (1 + F_m^{(2)}) \left(\frac{e}{m}\right)^2 \int_0^1 \rho \Psi^{(2)}(\rho)[J_1(me\rho) - \rho J_1(me)]\mathrm{d}\rho \right\}, \tag{56}$$

(III) Free edges at $|y| = b$ and $i = 3$;

$$B_m^{(3)} = \frac{4\nu\sinh\beta}{\pi^5 m^5 \bar{\Delta}_1} + \frac{\bar{\Delta}_4}{\bar{\Delta}_1} \left\{ \frac{G_m^{(3)}}{m^3} - (1 + F_m^{(3)}) \left(\frac{e}{m}\right)^2 \int_0^1 \rho \Psi^{(3)}(\rho)\,[J_1(me\rho) - \rho J_1(me)]\,\mathrm{d}\rho \right\}, \tag{57}$$

where

$$\bar{\Delta}_4 = 2 + (3 + \nu)\sinh^2\beta. \tag{58}$$

The remaining unknown constants for all cases can be obtained in the same form but with a different auxiliary function $\Psi^{(i)}(\rho)$ as follows:

$$A_m^{(i)} = -\frac{4}{\pi^5 m^5} + \left(\frac{e}{m}\right)^2 \int_0^1 \rho \Psi^{(i)}(\rho)[J_1(me\rho) - \rho J_1(me)]\mathrm{d}\rho; \quad i = 1, 2, 3, \tag{59}$$

$$C_m^{(i)} = -D_m^{(i)} = \frac{G_m^{(i)}}{m^3} - (1 + F_m^{(i)}) \left(\frac{e}{m}\right)^2 \int_0^1 \rho \Psi^{(i)}(\rho)[J_1(me\rho) - \rho J_1(me)]\mathrm{d}\rho; \quad i = 1, 2, 3. \tag{60}$$

The bending moments in the x- and y-directions of the scaled coordinates can be obtained from the expressions (see [13])

$$M_x = -D\left(\frac{\pi}{a}\right)^2 \left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \quad M_y = -D\left(\frac{\pi}{a}\right)^2 \left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right). \tag{61,62}$$

By an identical procedure to obtain the deflections in (53) and (54), the bending moments $M_x(x, 0)$, $M_y(x, 0)$, $M_x(\pi/2, y)$, and $M_y(\pi/2, y)$ can be expressed as

$$\frac{M_x(x, 0)}{qa^2\pi^2} = \sum_{m=1,3,5,\ldots}^{\infty} \nu S_m^{(i)} \sin(mx) + \int_0^1 L(x, \rho)\Psi^{(i)}(\rho)\mathrm{d}\rho; \quad 0 \le x \le \pi/2, \tag{63}$$

$$\frac{M_y(x, 0)}{qa^2\pi^2} = \sum_{m=1,3,5,\ldots}^{\infty} S_m^{(i)} \sin(mx) + \int_0^1 \bar{L}(x, \rho)\Psi^{(i)}(\rho)\mathrm{d}\rho; \quad 0 \le x \le \pi/2, \tag{64}$$

$$\frac{M_x(\pi/2, y)}{qa^2\pi^2} = \sum_{m=1,3,5,\ldots}^{\infty} \{4/\pi^5 m^3 + m^2 A_m^{(i)}(1-\nu)\cosh(my) + m^2 B_m^{(i)}[(1-\nu)my\sinh(my) - 2\nu\cosh(my)]$$
$$+m^2 C_m^{(i)}[(1+\nu)\sinh(my) - (1-\nu)my\cosh(my)]\}(-1)^{(m-1)/2}; \quad 0 \le y \le b, \tag{65}$$

$$\frac{M_y(\pi/2, y)}{qa^2\pi^2} = \sum_{m=1,3,5,\ldots}^{\infty} \{4\nu/\pi^5 m^3 - m^2 A_m^{(i)}(1-\nu)\cosh(my) - m^2 B_m^{(i)}[(1-\nu)my\sinh(my) + 2\cosh(my)]$$
$$+m^2 C_m^{(i)}[(1+\nu)\sinh(my) + (1-\nu)my\cosh(my)]\}(-1)^{(m-1)/2}; \quad 0 \le y \le b, \tag{66}$$

where

$$L(x,\rho) = -\frac{(1+\nu)xe\rho^2 H(e-x)}{2(e^2-x^2)^{1/2}} + \frac{(1+\nu)xeH(e\rho-x)}{2(e^2\rho^2-x^2)^{1/2}}$$
$$+2\nu e^2\rho \sum_{m=1,3,5,\ldots}^{\infty} H_m^{(i)}[J_1(me\rho) - \rho J_1(me)]\sin(mx), \tag{67}$$

$$\bar{L}(x,\rho) = -\frac{(1+\nu)xe\rho^2 H(e-x)}{2(e^2-x^2)^{1/2}} + \frac{(1+\nu)xeH(e\rho-x)}{2(e^2\rho^2-x^2)^{1/2}}$$
$$+2e^2\rho \sum_{m=1,3,5,\ldots}^{\infty} H_m^{(i)}[J_1(me\rho) - \rho J_1(me)]\sin(mx), \tag{68}$$

and the known functions $S_m^{(i)}$ and $H_m^{(i)}$ are differently defined for each case as in the following relations:

(I) Simply supported edges at $|y| = b$ and $i = 1$;

$$S_m^{(1)} = \frac{4}{\pi^5 m^3}\left(1 - \frac{2\mathrm{sech}\beta + \pi^5 m^2 G_m^{(1)}\tanh\beta}{2}\right), \tag{69}$$

$$1 + H_m^{(1)} = (1 + F_m^{(1)})\tanh\beta, \tag{70}$$

(II) Clamped edges at $|y| = b$ and $i = 2$;

$$S_m^{(2)} = \frac{4}{\pi^5 m^3}\left\{1 - \frac{4\mathrm{sech}\beta + \pi^5 m^2 G_m^{(2)}\tanh\beta}{2\Delta_1}\right\}, \tag{71}$$

$$\Delta_1(1 + H_m^{(2)}) = (1 + F_m^{(2)})\tanh\beta, \tag{72}$$

(III) Free edges at $|y| = b$ and $i = 3$;

$$S_m^{(3)} = \frac{4}{\pi^5 m^3}\left\{1 - \frac{4\nu\sinh\beta + \pi^5 m^2 G_m^{(3)}\bar{\Delta}_4}{2\bar{\Delta}_1}\right\}, \tag{73}$$

$$\bar{\Delta}_1(1 + H_m^{(3)}) = (1 + F_m^{(3)})\bar{\Delta}_4. \tag{74}$$

It should be noted that the bending moment in (63) and (64) can be derived with the help of the identity [10]

$$\sum_{m=1,3,5,\ldots}^{\infty} J_1(mt)\sin(mx) = \frac{1}{2}xt^{-1}(t^2 - x^2)^{-1/2}H(t-x); \quad x + t < \pi. \tag{75}$$

It is also noted that the first term of $L(x,\rho)$ in (67) and $\bar{L}(x,\rho)$ in (68) is singular at $x = e$ in the order of an inverse square root which agrees with the conclusion by Williams [4] for the vicinity of the juncture, and the remaining terms represent continuous functions in the region $0 \le x \le \pi/2$, provided that $e < \pi/2$. If $e < x \le \pi/2$, the singular part vanishes automatically because of the Heaviside function $H(e-x)$. This is one of the advantages of the present approach in that it deals with the singularity at the point of discontinuous support. In the Appendix, the method of isolating a singular dominant term in the bending moments $M_x(e-\varepsilon, 0)$ and $M_y(e-\varepsilon, 0)$ near the tip of a partial internal line support, where ε is the small distance measured from the singular point, is given. The bending-stress intensity factors (k) due to the singularity in the moments are also derived.

The corner forces of the plate can be evaluated from the relation

$$R = 2D(1-\nu)\left(\frac{\pi}{a}\right)^2 \frac{\partial^2 w}{\partial x \partial y}. \tag{76}$$

The corner forces are considered positive if they act on the plate in the downward direction in order to prevent the plate corners from rising during bending [13, Art. 22]. Therefore, the corner force at point $(0, b)$ of the plate can be determined by substituting $x = 0$ and $y = b$ in the deflection in (4) and then substituting in (76); this yields

$$\frac{R(0,b)}{qa^2\pi^2} = 2(1-\nu)\sum_{m=1,3,5,\ldots}^{\infty} [A_m^{(i)} \sinh\beta + B_m^{(i)}(\beta\cosh\beta + \sinh\beta) - C_m^{(i)}\sinh\beta]m^2. \tag{77}$$

It is interesting to note that, since the singularity at the end of the partial internal line support is of $O(\varepsilon^{-1/2})$ in the moment, the shear singularity at the end of the partial internal support is in the order of $O(\varepsilon^{-3/2})$. The shear distribution along the partial internal line support is not integrable and the total shearing force transmitted to the partial internal line support cannot be computed directly. However, it has been shown that the shear distribution derived by Yang [9] is of $O(\varepsilon^{-1/2})$ which is integrable and in disagreement with the present results and the works by Williams [4], Grigolyuk and Tolkachev [15, Sect. 8.7], and Monegato and Strozzi [16]. Although the shear distribution is nonintegrable, the reaction force of the partial internal line support can still be determined from the equilibrium condition of the plate.

4 Results and discussions

Simpson's rule was used to transform the Fredholm integral equation of (41) into a system of linear algebraic equations which can be solved numerically for the discretized value of the auxiliary function $\Psi^{(i)}(\rho)$ by using Gaussian elimination with partial pivoting [17, Sect. 3.4]. The improper infinite integral in the kernel $K(\rho, r)$ in (44) was evaluated numerically by the 16-point Gauss–Legendre quadrature formula [18, Sect. 25.4.29]. The infinite series in $K(\rho, r)$ in (44), $f(\rho)$ in (45) and other physical quantities were calculated to a relative error of 0.00001.

The numerical results were obtained for a scaled square plate of π with different support conditions at $|y| = \pi/2$ and the Poisson ratio is taken as 0.3. The physical quantities of plates with and without a partial internal line support were illustrated.

It is observed that, for the case of free edges at $|y| = \pi/2$, the auxiliary function $\Psi^{(3)}(\rho)$ is dependent on the value of Poisson's ratio. This can be seen in the functions $F_m^{(3)}$ in (27) and $G_m^{(3)}$ in (28). Thus, the normalized deflections of the plate with a partial internal line support and having simply supported and clamped edges at $|y| = \pi/2$ as shown in Fig. 2 are valid for all values of the Poisson ratio. The deflections $w(x, 0)$ along $0 \le x \le e$ of the plates are small for $e/\pi < 0.35$. The deflections $w(\pi/2, y)$ for $0.05 < e/\pi < 0.45$ are quite close for all cases.

The bending moments depend on the Poisson ratio as shown in (61) and (62). Thus, all results shown in Figs. 3 and 4 are for a Poisson ratio of 0.3. The bending moments $M_x(x, 0)$ and $M_y(x, 0)$ along $0 \le x \le e$ for plates having a partial internal line support are shown in Fig. 3, and they become singular according to an inverse square root at $x = e$ as indicated in (67) and (68). This singular behavior did not appear in [9]; the bending moments have finite values which disagrees with Williams's work [4]. For plates without a partial internal line support, the bending moments are regular along the line $0 \le x \le \pi/2$, $y = 0$ and have a maximum value at $x = \pi/2$. In Fig. 4, the bending moments $M_x(\pi/2, y)$ and $M_y(\pi/2, y)$ of all plates show a similar trend. It is noted that the bending moments $M_x(\pi/2, y)$ and $M_y(\pi/2, y)$ for $0.05 < e/\pi < 0.35$ are almost the same.

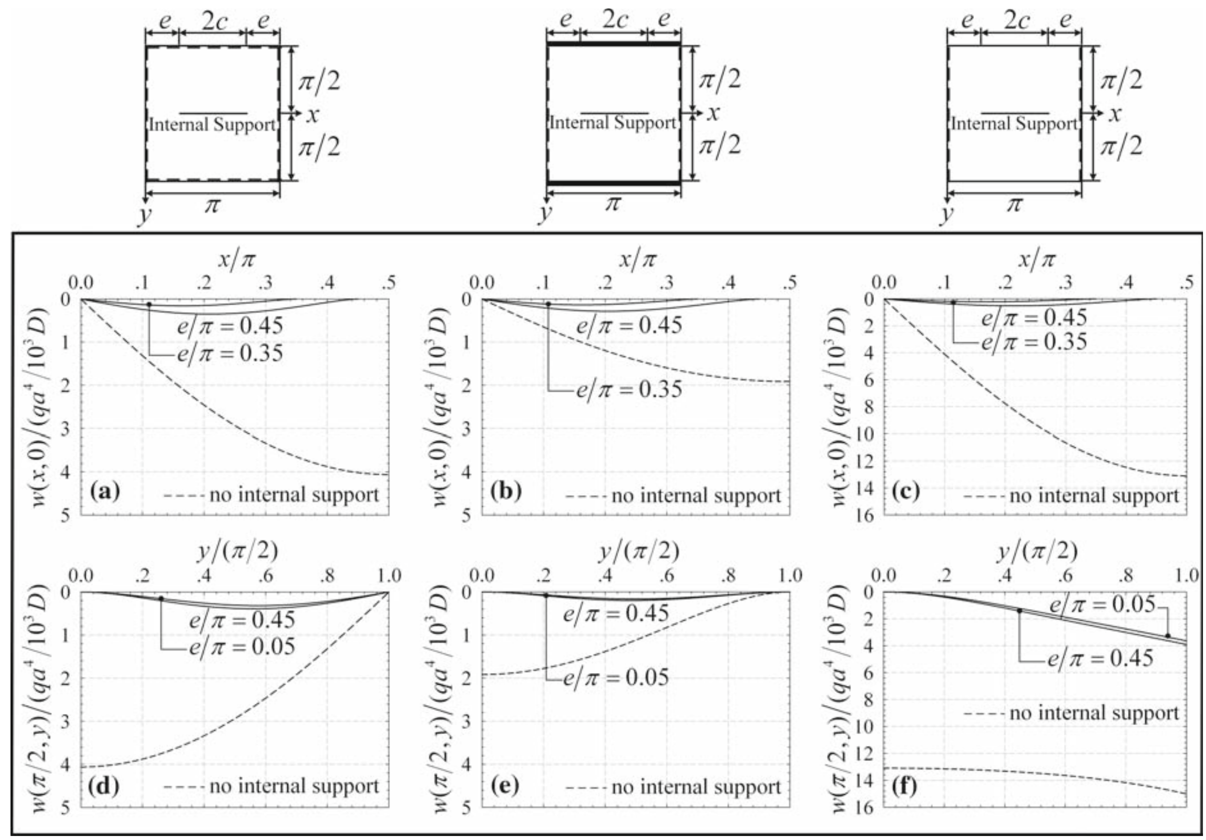

Fig. 2 Deflections $w(x, 0)$ and $w(\pi/2, y)$ for square plate with a partial internal line support

The resultant forces exerted by the supports and the corner forces are calculated and shown in Fig. 5. It is seen that corner forces exist only for plates having simply supported and free edges at $|y| = \pi/2$, but not for plates with clamped edges at $|y| = \pi/2$ due to the absence of twisting moments at the two adjacent edges of the corner. The positive sign of the corner force represents the downward direction to prevent the corner of the plate from moving up during bending [13, Art. 22]; therefore the directions of the corner force $R(0, \pi/2)$ for plates having simply supported and free edges at $|y| = \pi/2$ are downward and upward, respectively. The corner force is dependent on the ratio of $2c/\pi$. When $2c/\pi$ increases, the magnitude of $R(0, \pi/2)$ decreases for plates having simply supported edges at $|y| = \pi/2$ and increases for plates with free edges at $|y| = \pi/2$ as shown in Fig. 5. It is observed that the reaction force of the partial internal line support of the plate in all cases increases as the length of the partial internal line support increases and the reaction forces at the boundary of the plate decrease.

Figure 6 illustrates the bending-stress intensity factor (k) for three cases of a plate having a partial internal line support. It is noted that the bending-stress intensity factor in the x- and y-directions is the same as shown in the Appendix. The values of k are all zero when $2c = \pi$ corresponding to a plate having a full-length internal line support or a continuous plate. The variation of k is that it increases monotonically and reaches a maximum for a certain length of the partial internal line support, and then decreases when the length of the partial internal line support approaches zero ($2c \to 0$), which corresponds to the plate being supported by a column at the center. It is noted that the maximum value of k occurs when $2c = 0.06\pi, 0.08\pi, 0.06\pi$ for plates having simply supported, clamped, and free edges at $|y| = \pi/2$, respectively. This result has a similar maximum bending-stress intensity-factor trend in comparison with the work by Stahl and Keer [19] for the case of a simply supported, free circular plate.

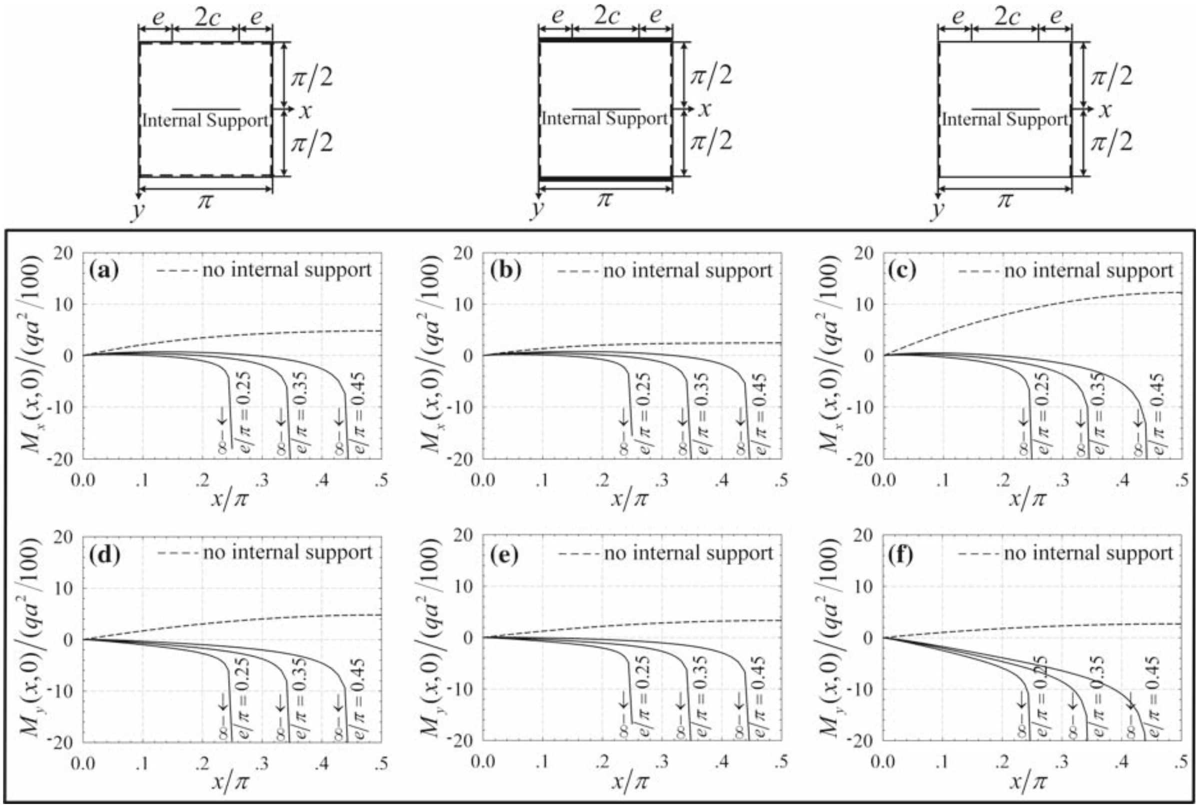

Fig. 3 Moments $M_x(x, 0)$ and $M_y(x, 0)$ for square plate with a partial internal line support

5 Conclusions

The problem of rectangular plates with a partial internal line support, simply supported on two opposite edges, and the other two edges having three different boundary conditions representing a uniformly distributed load, has been considered. Since the plate has mixed conditions along the partial internal line support, the problem could be formulated through dual-series equations. A moment singularity of square-root type has been taken into account in the analysis at the tips of the partial internal line support. Based on a method involving finite Hankel transforms, together with the proper singularity, the dual-series equations were reduced to an inhomogeneous Fredholm integral equation of the second kind in terms of the unknown auxiliary function. The integral equation could be solved numerically to determine the unknown auxiliary function by using a standard numerical technique. The deflections, bending moments, support reactions, and corner forces of the plates could be obtained in closed form. Results were presented for square plates. The bending-stress intensity factor at the tips of the partial internal line support has also been determined. In addition, the solution of a plate supported by a column at the center was obtained as a limiting case when the length of the partial internal line approaches zero ($2c \to 0$). For plates without a partial internal line support ($2c = 0$), the solution could be obtained by applying some modifications to the solution for plates with partial internal line support and the numerical results were also obtained for a square plate. It is noted that the method used in this paper is found to be efficient for obtaining the exact closed-form solution for rectangular plates with mixed conditions, subjected to a uniformly distributed load. It could be extended to solve the problem of advancing contact between a rectangular plate and a sagged partial internal line support, or the problem of a receding contact between a rectangular plate and unilateral supports. An inverse-square-root shear singularity must be introduced instead of the inverse-square-root moment singularity at the point of discontinuity of the boundary conditions for the mentioned free-contact problems.

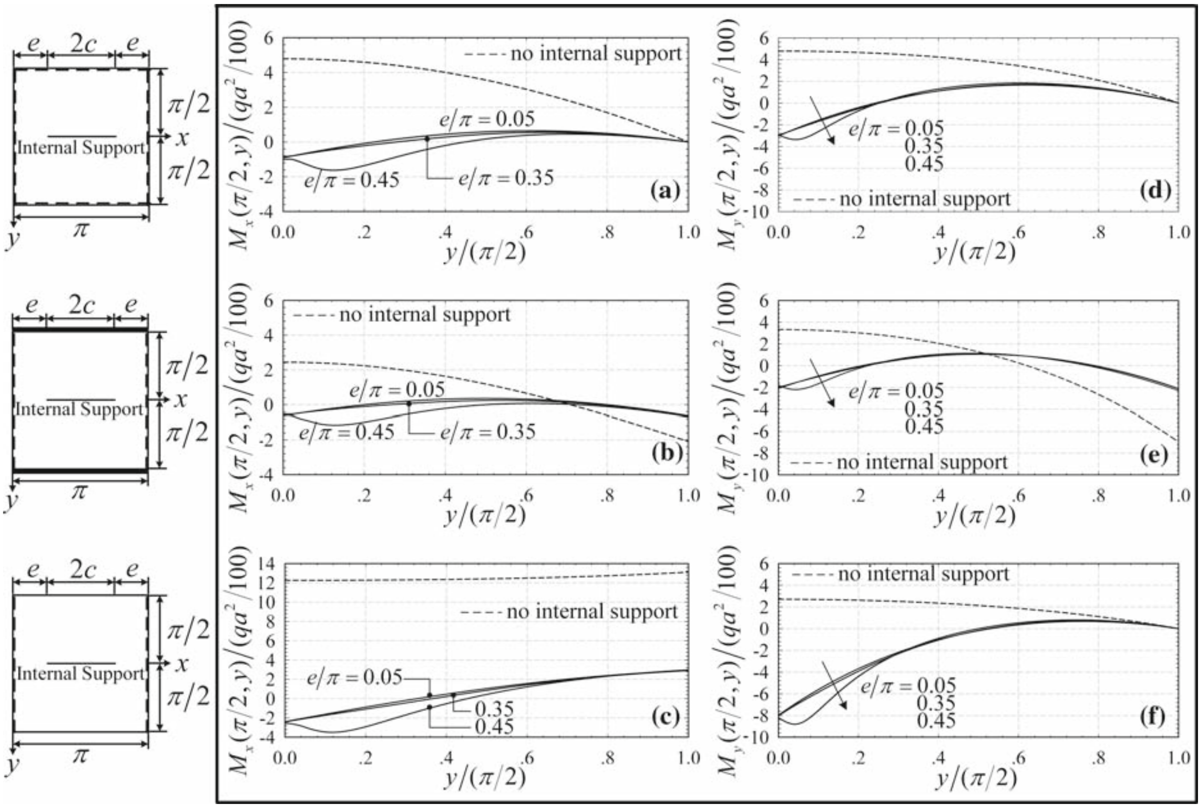

Fig. 4 Moments $M_x(\pi/2,\, y)$ and $M_y(\pi/2,\, y)$ for square plate with a partial internal line support

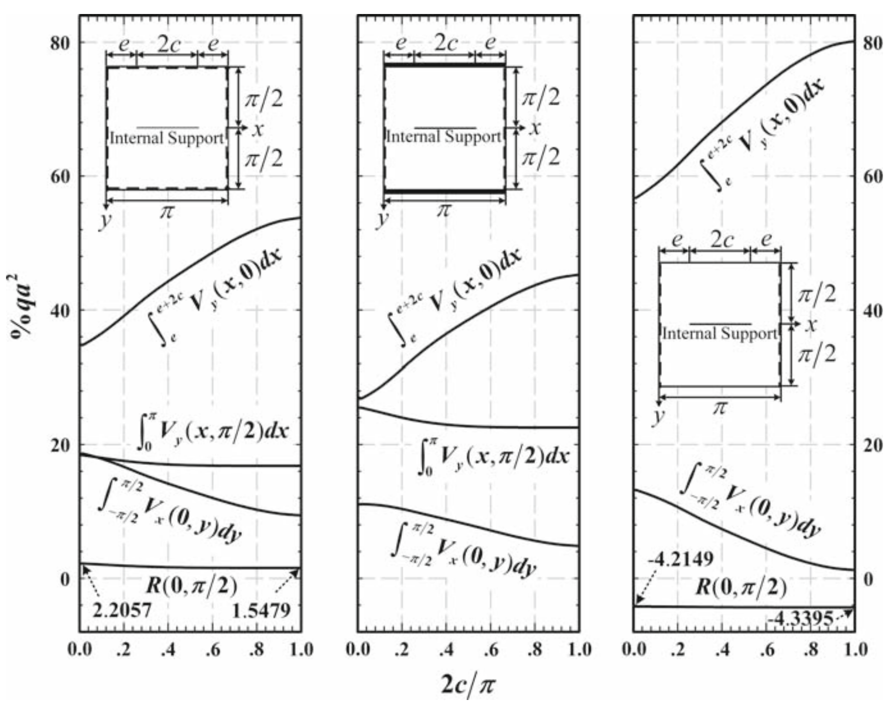

Fig. 5 Proportion of the resultant forces resisted by the supports of square plate

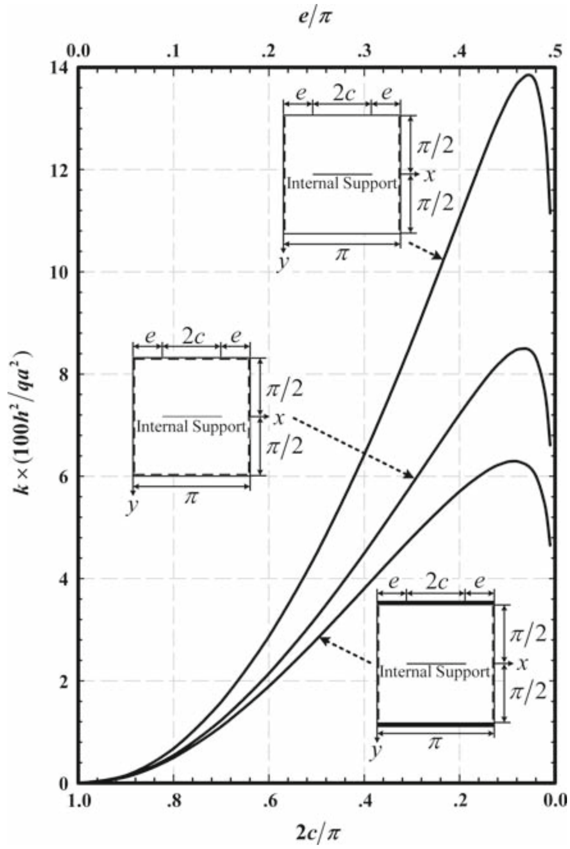

Fig. 6 Bending-stress intensity factors for square plate

Appendix: Bending-stress intensity factor

In order to derive the bending-stress intensity factor of a plate having partial internal line support, the dominant singular term of the moment field is isolated.

For the present work, the singularity in the bending moments M_x and M_y in the vicinity of the tip of the partial internal support ($x \to e$, $y = 0$) is taken as an inverse-square-root function. The verification is made by setting $y = 0$ in (4) and then substituting in (61) and (62), respectively, thus

$$\frac{M_x(x,0)}{qa^2\pi^2} = \sum_{m=1,3,5,\ldots}^{\infty} [4/\pi^5 m^3 + (1-\nu)m^2 A_m^{(i)} - 2\nu m^2 B_m^{(i)}] \sin(mx), \tag{78}$$

$$\frac{M_y(x,0)}{qa^2\pi^2} = \sum_{m=1,3,5,\ldots}^{\infty} [4\nu/\pi^5 m^3 - (1-\nu)m^2 A_m^{(i)} - 2m^2 B_m^{(i)}] \sin(mx). \tag{79}$$

Using $A_m^{(i)}$ in (15), (21), (29), $B_m^{(i)}$ in (16), (22), (30), and $P_m^{(i)}$ in (12), (18), (26) for each case, Eqs. (78) and (79) can be written as

(I) Simply supported edges at $|y| = b$ and $i = 1$;

$$\frac{M_x(x,0)}{qa^2\pi^2} = (1+\nu) \sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(1)} \sin(mx) + (1+\nu) \sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(1)} Q_m^{(1)} \sin(mx)$$
$$+ \sum_{m=1,3,5,\ldots}^{\infty} \left\{ \frac{4(\cosh\beta - 1) - 2(1-\nu)\beta\tanh\beta}{\pi^5 m^3 \cosh\beta} - \frac{G_m^{(1)}}{m} \left[\frac{(1-\nu)}{(1+F_m^{(1)})} + 2\nu\tanh\beta \right] \right\} \sin(mx), \tag{80}$$

$$\frac{M_y(x,0)}{qa^2\pi^2} = (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(1)} \sin(mx) + (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(1)} \bar{Q}_m^{(1)} \sin(mx)$$
$$+ \sum_{m=1,3,5,\ldots}^{\infty} \left\{ \frac{4\nu(\cosh\beta - 1) + 2(1-\nu)\beta\tanh\beta}{\pi^5 m^3 \cosh\beta} + \frac{G_m^{(1)}}{m}\left[\frac{(1-\nu)}{(1+F_m^{(1)})} - 2\tanh\beta\right]\right\} \sin(mx), \quad (81)$$

where

$$1 + Q_m^{(1)} = \frac{(1+\nu)\sinh\beta\cosh\beta - (1-\nu)\beta}{(1+\nu)(\sinh\beta\cosh\beta - \beta)}, \quad (82)$$

$$1 + \bar{Q}_m^{(1)} = \frac{(1+\nu)\sinh\beta\cosh\beta + (1-\nu)\beta}{(1+\nu)(\sinh\beta\cosh\beta - \beta)}, \quad (83)$$

(II) Clamped edges at $|y| = b$ and $i = 2$;

$$\frac{M_x(x,0)}{qa^2\pi^2} = (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(2)} \sin(mx) + (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(2)} Q_m^{(2)} \sin(mx)$$
$$+ \sum_{m=1,3,5,\ldots}^{\infty} \left\{ \frac{4[\sinh\beta\cosh\beta + \beta - (1+\nu)\sinh\beta - (1-\nu)\beta\cosh\beta]}{\pi^5 m^3(\sinh\beta\cosh\beta + \beta)}\right.$$
$$\left. - \frac{G_m^{(2)}}{m}\left[\frac{(1-\nu)}{(1+F_m^{(2)})} + \frac{2\nu\tanh\beta}{\Delta_1}\right]\right\} \sin(mx), \quad (84)$$

$$\frac{M_y(x,0)}{qa^2\pi^2} = (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(2)} \sin(mx) + (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(2)} \bar{Q}_m^{(2)} \sin(mx)$$
$$+ \sum_{m=1,3,5,\ldots}^{\infty} \left\{ \frac{4[\nu(\sinh\beta\cosh\beta + \beta) - (1+\nu)\sinh\beta + (1-\nu)\beta\cosh\beta]}{\pi^5 m^3(\sinh\beta\cosh\beta + \beta)}\right.$$
$$\left. + \frac{G_m^{(2)}}{m}\left[\frac{(1-\nu)}{(1+F_m^{(2)})} - \frac{2\tanh\beta}{\Delta_1}\right]\right\} \sin(mx), \quad (85)$$

where

$$1 + Q_m^{(2)} = \frac{(1+\nu)\tanh\beta + (1-\nu)\beta^2(\tanh\beta - \coth\beta)}{(1+\nu)\tanh\beta + (1+\nu)\beta^2(\tanh\beta - \coth\beta)}, \quad (86)$$

$$1 + \bar{Q}_m^{(2)} = \frac{(1+\nu)\tanh\beta - (1-\nu)\beta^2(\tanh\beta - \coth\beta)}{(1+\nu)\tanh\beta + (1+\nu)\beta^2(\tanh\beta - \coth\beta)}, \quad (87)$$

(III) Free edges at $|y| = b$ and $i = 3$;

$$\frac{M_x(x,0)}{qa^2\pi^2} = (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(3)} \sin(mx) + (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(3)} Q_m^{(3)} \sin(mx)$$
$$+ \sum_{m=1,3,5,\ldots}^{\infty} \left\{ \frac{4[(3+\nu)\sinh\beta\cosh\beta - (1-\nu)\beta(1+\nu\cosh\beta) + (1-\nu)\nu\sinh\beta]}{\pi^5 m^3 \bar{\Delta}_1}\right.$$
$$\left. - \frac{G_m^{(3)}}{m}\left[\frac{(1-\nu)}{(1+F_m^{(3)})} + \frac{2\nu[2 + (3+\nu)\sinh^2\beta]}{\bar{\Delta}_1}\right]\right\} \sin(mx), \quad (88)$$

$$\frac{M_y(x,0)}{qa^2\pi^2} = (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(3)}\sin(mx) + (1+\nu)\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m^{(3)}\bar{Q}_m^{(3)}\sin(mx)$$
$$+\sum_{m=1,3,5,\ldots}^{\infty}\left\{\frac{4\nu[(3+\nu)\sinh\beta\cosh\beta-(1-\nu)\beta(1-\cosh\beta)-(3+\nu)\sinh\beta]}{\pi^5 m^3\bar{\Delta}_1}\right.$$
$$\left.+\frac{G_m^{(3)}}{m}\left[\frac{(1-\nu)}{(1+F_m^{(3)})}-\frac{2[2+(3+\nu)\sinh^2\beta]}{\bar{\Delta}_1}\right]\right\}\sin(mx), \tag{89}$$

where

$$1+Q_m^{(3)} = \frac{(1+\nu)[1-\nu+(3+\nu)\cosh^2\beta]+(1-\nu)^2\beta^2}{(1+\nu)[(1-\nu)\beta^2+(1+\nu)\eta''+(3+\nu)\cosh^2\beta]}, \tag{90}$$

$$1+\bar{Q}_m^{(3)} = \frac{(1+\nu)(3+\nu)\sinh^2\beta-(1-\nu)^2\beta^2}{(1+\nu)[(1-\nu)\beta^2+(1+\nu)\eta''+(3+\nu)\cosh^2\beta]}. \tag{91}$$

It is noted that the weight functions $Q_m^{(i)}$ in $M_x(x,0)$ and $\bar{Q}_m^{(i)}$ in $M_y(x,0)$ approach zero as $m\to\infty$. Thus, only the first series in $M_x(x,0)$ and $M_y(x,0)$ is the dominant singular part and needs to be considered. Substituting $P_m^{(i)}$ of (36) in the first series and using the identity given in (75) together with introducing $t=e\rho$ and $x=e-\varepsilon$, the dominant singular part of the bending moments can be expressed as

$$M_x(e-\varepsilon,0)=M_y(e-\varepsilon,0)=M(e-\varepsilon,0)=-\frac{(1+\nu)qa^2\pi^2e^{3/2}}{2(2\varepsilon)^{1/2}}\int_0^1\rho^2\Psi^{(i)}(\rho)\mathrm{d}\rho, \tag{92}$$

where $0<\varepsilon\ll 1$.

The bending stresses σ_x and σ_y are given by

$$\sigma_x = 12M_x\delta/h^3,\quad \sigma_y = 12M_y\delta/h^3, \tag{93,94}$$

in which δ is the coordinate measured from and perpendicular to the middle plane of the plate.

In the vicinity of the tip of the partial internal line support, the bending stresses σ_x and σ_y can be written in the form of bending-stress intensity factors k_x and k_y, respectively,

$$\sigma_x = k_x/\varepsilon^{1/2},\quad \sigma_y = k_y/\varepsilon^{1/2}. \tag{95,96}$$

Substituting (92) in (93), (94) and comparing with (95), (96) yields

$$k_x=k_y=k=-\frac{6(1+\nu)qa^2\pi^2e^{3/2}\delta}{\sqrt{2}h^3}\int_0^1\rho^2\Psi^{(i)}(\rho)\mathrm{d}\rho. \tag{97}$$

Considering the bending-stress intensity factor at the top surface of the plate, and substituting $\delta=-h/2$ in (97) yields

$$k=\frac{3(1+\nu)qa^2\pi^2e^{3/2}}{\sqrt{2}h^2}\int_0^1\rho^2\Psi^{(i)}(\rho)\mathrm{d}\rho. \tag{98}$$

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